Fixed Points of the (n, 3)-Josephus Permutations

Sarah Breede[1] and Christy Finch[2]

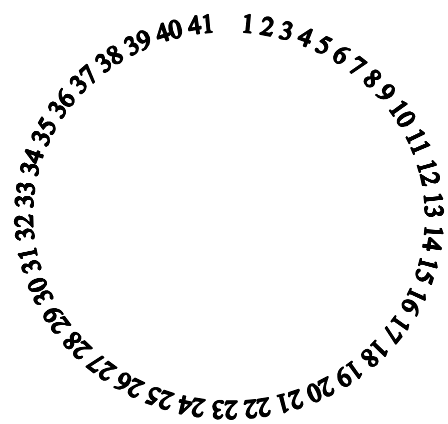
July 2001

Abstract

In this paper, a variation of the classical Josephus Dilemma is considered. In a circle of n men when we eliminate every third person, the order of elimination defines an *(n,3)*-Josephus permutation. We discover and prove fifteen remarkable patterns in the fixed points of these permutations.

1 Introduction

Flavius Josephus was a noted historian and military general who lived in Greece during the first century A. D. In his writings [5], Josephus recounts a battle in Galilee where the Roman army led by Vespasian seized the city of Jotapata. There, Josephus and 40 of his men were captured. Instead of falling into the fate of slavery, his men considered committing suicide. However, Josephus was against this idea so he devised a way for each man to die by the hands of another until only one person remained to kill himself. The process was simple: Josephus had his men arrange themselves in a circle and each man was assigned to a number from 1 to 41.



Person one was instructed to kill person two, person three to kill person four, person five to kill person six and this process was to continue around the circle until all were eliminated. In the first pass around the circle, the even-numbered men died. Then person forty-one eliminated person one and the process continued until person nineteen was the last person alive. Before the process began, Josephus ensured that he would be this last person remaining. Evidently, he escaped the duty of killing himself so he could tell the tale we know today.

[1] Department of Mathematics, Kennesaw State University, 1000 Chastain Rd Kennesaw, GA 30144
sbreede@hotmail.com
[2] Department of Mathematics, Centenary College, 2911 Centenary Blvd., Shreveport, LA 71134
cfinch@centenary.edu

Given any number of people in the circle, who would survive? This question has gained fame within the mathematical community and is commonly called the Josephus Dilemma. In [2], Leonhard Euler (1707-1783) examined patterns that are found in various orders of elimination. In [4], for different values of *n*, Herstein et al. listed the orders of elimination as permutations and examined their cyclic nature. In the midst of what seemed to be chaos, hidden structures were found. More recently, Cook et al. conducted an examination on the “Josephus permutations” in [1]. Before we describe their results, we will introduce and explain some definitions that will help us generalize the Josephus Dilemma.

Definition 1.1: We define the function $J(n, k)$ to be the last person alive in a circle of n people where every k^{th} person is eliminated.

Definition 1.2: The variable k shall be called the *skip factor*.

Definition 1.3: For each $n \in \mathbf{N}$, there is a one-to-one correspondence, written as $J_{n,k}$, from the set $\{1,2,3,...,n\}$ onto itself where $J_{n,k}(p) = j$ if person j is the p^{th} person eliminated. This correspondence is called a *(n, k)-Josephus permutation*.

Definition 1.4: We call p a *fixed point* of $J_{n,k}$ if $J_{n,k}(p) = p$.

In the original Josephus Dilemma where $n = 41$ and $k = 2$, $J(41,2) = 19$. Below are a few more examples of (n, k)-Josephus permutations with various values of n and k.

Table 1

$n = 9 \quad k = 2$

Order Eliminated	1	2	3	4	5	6	7	8	9
Person Eliminated	2	4	6	8	1	5	9	7	3

Table 2

$n = 20 \quad k = 5$

Order Eliminated	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Person Eliminated	5	10	15	20	6	12	18	4	13	1	9	19	11	3	17	16	2	8	14	7

Table 3

$n = 24 \quad k = 3$

Order Eliminated	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Person Eliminated	3	6	9	12	15	18	21	24	4	8	13	17	22	2	10	16	23	7	19	5	20	14	1	11

We find fixed points in Tables 2 and 3 where $J_{20,5}(16)=16$, $J_{24,3}(16)=16$ and $J_{24,3}(19)=19$. However, Table 1 shows that the order of elimination, $J_{9,2}$, does not have a fixed point.

Expanding on the idea of a fixed point, we can say that if *p* is a fixed point, then in the order of elimination of *n* men where every k^{th} person is eliminated, *p* is the p^{th} person eliminated. In Table 1, we see the order of elimination for 9 men where every 2^{nd} person is eliminated. This (9, 2)-Josephus permutation can be written in tableau and cyclic notation as follows

$$J_{9,2}=\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\2\,4\,6\,8\,1\,5\,9\,7\,3\end{pmatrix}=(1\ 2\ 4\ 8\ 7\ 9\ 3\ 6\ 5)$$

We see that the (9, 2)-Josephus permutation is a 9-cycle in cyclic notation. In Table 3, we see a different cyclic structure. It can be written as follows

$$J_{24,3}=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24\\3&6&9&12&15&18&21&24&4&8&13&17&22&2&10&16&23&7&19&5&20&14&1&11\end{pmatrix}$$
$$=(1\ 3\ 9\ 4\ 12\ 17\ 23)(2\ 6\ 18\ 7\ 21\ 20\ 5\ 15\ 10\ 8\ 24\ 11\ 13\ 22\ 14)(16)(19).$$

This permutation has one 7-cycle, one 15-cylcle and two 1-cycles. These 1-cycles are the fixed points. If a permutation can be written as a product of disjoint cycles with at least one 1-cycle, then the permutation contains a fixed point.

In [1], Cook et al. give a thorough description of the fixed points of $J_{n,2}$ where $n=2^m$. We know that with a skip factor of two, half of the people are eliminated in each cycle of elimination. Because of this, they prove that *p* is a fixed point of $J_{n,2}$ if and only if $m=\frac{(2n+1)(2^{b-1}-1)}{2^b-1}$ for $b>1$. Does a similar formula result when we choose a skip factor of three?

With this question in mind, we initially used the same methods found in [1] to determine a formula for the fixed points of $J_{n,3}$ for $n=3^m$. However, with a skip factor of three, we know that approximately one-third of the people are eliminated in each cycle yet two-thirds remain. Because of this, a nice formula did not surface. Escaping this approach, we looked at the (*n*, *3*)-Josephus permutations tabulated as sequences for *n*<36. Table 4 illustrates these sequences with the order of elimination increasing across the *x*-axis and the values of *n* increasing down the *y*-axis.

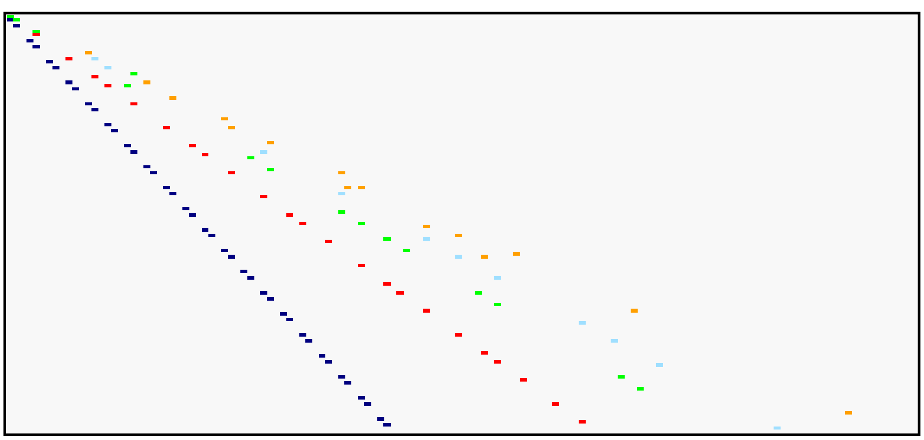
Table 4

Order Eliminated

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
1	1																																		
2	1	2																																	
3	3	1	2																																
4	3	2	4	1																															
5	3	1	5	2	4																														
6	3	6	4	2	5	1																													
7	3	6	2	7	5	1	4																												
8	3	6	1	5	2	8	4	7																											
9	3	6	9	4	8	5	2	7	1																										
10	3	6	9	2	7	1	8	5	10	4																									
11	3	6	9	1	5	10	4	11	8	2	7																								
12	3	6	9	12	4	8	1	7	2	11	5	10																							
13	3	6	9	12	2	7	11	4	10	5	1	8	13																						
14	3	6	9	12	1	5	10	14	7	13	8	4	11	2																					
15	3	6	9	12	15	4	8	13	2	10	1	11	7	14	5																				
16	3	6	9	12	15	2	7	11	16	5	13	4	14	10	1	8																			
17	3	6	9	12	15	1	5	10	14	2	8	16	7	17	13	4	11																		
18	3	6	9	12	15	18	4	8	13	17	5	11	1	10	2	16	7	14																	
19	3	6	9	12	15	18	2	7	11	16	1	8	14	4	13	5	19	10	17																
20	3	6	9	12	15	18	1	5	10	14	19	4	11	17	7	16	8	2	13	20															
21	3	6	9	12	15	18	21	4	8	13	17	1	7	14	20	10	19	11	5	16	2														
22	3	6	9	12	15	18	21	2	7	11	16	20	4	10	17	1	13	22	14	8	19	5													
23	3	6	9	12	15	18	21	1	5	10	14	19	23	7	13	20	4	16	2	17	11	22	8												
24	3	6	9	12	15	18	21	24	4	8	13	17	22	2	10	16	23	7	19	5	20	14	1	11											
25	3	6	9	12	15	18	21	24	2	7	11	16	20	25	5	13	19	1	10	22	8	23	17	4	14										
26	3	6	9	12	15	18	21	24	1	5	10	14	19	23	2	8	16	22	4	13	25	11	26	20	7	17									
27	3	6	9	12	15	18	21	24	27	4	8	13	17	22	26	5	11	19	25	7	16	1	14	2	23	10	20								
28	3	6	9	12	15	18	21	24	27	2	7	11	16	20	25	1	8	14	22	28	10	19	4	17	5	26	13	23							
29	3	6	9	12	15	18	21	24	27	1	5	10	14	19	23	28	4	11	17	25	2	13	22	7	20	8	29	16	26						
30	3	6	9	12	15	18	21	24	27	30	4	8	13	17	22	26	1	7	14	20	28	5	16	25	10	23	11	2	19	29					
31	3	6	9	12	15	18	21	24	27	30	2	7	11	16	20	25	29	4	10	17	23	31	8	19	28	13	26	14	5	22	1				
32	3	6	9	12	15	18	21	24	27	30	1	5	10	14	19	23	28	32	7	13	20	26	2	11	22	31	16	29	17	8	25	4			
33	3	6	9	12	15	18	21	24	27	30	33	4	8	13	17	22	26	31	2	10	16	23	29	5	14	25	1	19	32	20	11	28	7		
34	3	6	9	12	15	18	21	24	27	30	33	2	7	11	16	20	25	29	34	5	13	19	26	32	8	17	28	4	22	1	23	14	31	10	
35	3	6	9	12	15	18	21	24	27	30	33	1	5	10	14	19	23	28	32	2	8	16	22	29	35	11	20	31	7	25	4	26	17	34	13

Upon highlighting the fixed points of each sequence, we noticed several patterns in the table worth investigating. Curious to see more data, we examined the (*n*, *3*)-Josephus permutations tabulated as sequences for $n<140$ and found remarkable rays of fixed points that encouraged us to seek some generality. Table 5 illustrates these patterns below.

Table 5



Notice the left-most diagonal of highlighted points. We were able to describe the values of n that contained the fixed points of this diagonal as the recursive formula

$$n_i = \begin{cases} 2 & if \ i = 1 \\ 4 & if \ i = 2 \\ n_{i-2} + 7 & if \ i > 2 \end{cases}$$

and we were able to identify the fixed points corresponding to each n_i with the recursive formula

$$p_i = \begin{cases} 1 & if \ i = 1 \\ 2 & if \ i = 2 \\ p_{i-2} + 3 & if \ i > 2. \end{cases}$$

This relationship is easy to prove by mathematical induction. We shall note that these fixed points occur in the second pass through the circle.

Definition 1.5: Each pass through the circle is called a *round*.

Examining Table 5 more closely, we note two more distinct diagonals of fixed points that lie in the third and fourth rounds. We shall describe and prove these patterns of fixed points in the following sections. But wait, is there a pattern in the first round? Are there fixed points? Well, since the skip factor of our (n, 3)-Josephus permutations is 3, we know that in the first pass through the circle, all multiples of three are eliminated. Person 3 is the first to be eliminated, person 6 is the second to be eliminated, person 9 is the third to be eliminated, and so forth. So it appears that a multiple of three cannot be a fixed point. This leads us into our first lemma.

Lemma 1.6: *If* $J_{n,3}(p) = p$ *then* $3 \nmid p$*. Thus, any fixed point has the form* $p = 3m + 1$ *or* $p = 3m + 2$ *for some non-negative integer m.*

Proof: This proof follows easily by contradiction. Assume $J_{n,3}(p) = p$ and $3 \mid p$. Then, $p = 3q$ for some non-negative integer q. Therefore, person p is the q^{th} person to be eliminated so we have $J_{n,3}(q) = p$. But, since $q = \frac{p}{3} \neq p$, then $J_{n,3}(p) \neq p$ and this results in a contradiction; p is not divisible by 3. Therefore, any fixed point must be of the form $p = 3m + 1$ or $p = 3m + 2$ for some non-negative integer m. □

Now consider how we arrive at any fixed point within a sequence of elimination. We must count how many people are killed in each pass through the circle in order to determine if p is the p^{th} person killed. Since the skip factor is three, approximately one-third of n people are eliminated in the first round. We define the variable d_i to be the number of people eliminated in the i^{th} round and n_i to be the number of people that remain. So, in the first pass, $d_1 = \left\lfloor \frac{n}{3} \right\rfloor$. We use the notation $\left\lfloor \frac{n}{3} \right\rfloor$ to indicate the greatest integer value less than or equal to $\frac{n}{3}$. In each pass i through the circle, $d_i = \left\lfloor \frac{n_{i-1}}{3} \right\rfloor$ for i greater than one. However, the value of d_i depends on whether $n_i \equiv 0 \bmod 3$, $n_i \equiv 1 \bmod 3$, or $n_i \equiv 2 \bmod 3$.

Lemma 1.7: *Let* $n_i \equiv a \bmod 3$. *Then,* $\left\lfloor \frac{n_i}{3} \right\rfloor = \frac{n_i - a}{3}$.

Proof: Assume $n_i \equiv a \bmod 3$. Since $n_i \equiv a \bmod 3$, then 3 divides $n_i - a$. Thus,

$$\left\lfloor \frac{n_i}{3} \right\rfloor = \left\lfloor \frac{n_i - a + a}{3} \right\rfloor = \left\lfloor \frac{(n_i - a) + a}{3} \right\rfloor = \left\lfloor \frac{(n_i - a)}{3} + \frac{a}{3} \right\rfloor = \frac{n_i - a}{3} + \left\lfloor \frac{a}{3} \right\rfloor.$$

Since $n_i \equiv a \bmod 3$, then a is 0, 1, or 2. So, $\left\lfloor \frac{a}{3} \right\rfloor = 0$. Therefore, $\left\lfloor \frac{n_i}{3} \right\rfloor = \frac{n_i - a}{3}$. □

We will now discuss the patterns of fixed points for rounds 2, 3 and 4.

2 Fixed Points in Round 2

Beginning with the sequence

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, \ldots\}$$

of men, we are left with the following sequence after the first cycle of elimination

$$\alpha_j = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, \ldots\}.$$

This sequence can be written as the following piecewise function for $x \geq 0$.

$$\alpha_j = \begin{cases} \frac{1}{2}(3x - 1) \text{ if } j = 2x + 1 \\ \frac{1}{2}(3x - 2) \text{ if } j = 2x \end{cases}$$

Theorem 2.1: *If m is a non-negative integer then* $J_{7m+2,3}(3m+1) = 3m+1$.

Proof: Let $n = 7m + 2$. There are three possibilities for the congruence of n modulo 3: $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. We shall generalize these three cases by letting $a \in \{1,2,3\}$.

Suppose $n \equiv a \bmod 3$. We define d_i to be the number of people killed during the i^{th} cycle of elimination. Thus,

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n - a}{3} = \frac{7m + 2 - a}{3}.$$

In the second cycle of elimination, we know that we will eliminate at least $3m + 1$ people. We want to show that the $(3m + 1)$-*st* person killed is person $3m + 1$. Let z be the number of men that must be eliminated in the second cycle in order to eliminate exactly $3m+1$ people. We calculate z the following way:

$$z = 3m + 1 - d_1 = 3m + 1 - \frac{7m + 2 - a}{3} = \frac{2m + 1 + a}{3}.$$

Using our closed formula for α_j, we want to determine the value of α_{3z-a}. We start by calculating 3z – *a*.

$$3z - a = 3\left(\frac{2m + 1 + a}{3}\right) - a = 2m + 1.$$

So,

$$\alpha_{3z-a} = \frac{1}{2}\big(3(2m + 1) - 1\big) = 3m + 1.$$

Therefore, $J_{7m+2,3}(3m + 1) = 3m + 1$. □

Theorem 2.2: *If m is a non-negative integer then* $J_{7m+4,3}(3m + 2) = 3m + 2$.

Proof: Let $n = 7m + 4$ and $a = 1,2$, or 3. Suppose $n \equiv a \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n - a}{3} = \frac{7m + 4 - a}{3}.$$

In the second cycle of elimination, we know that we will eliminate at least 3*m* + 2 people. We want to show that the (3*m* + 2)-*nd* person killed is person 3*m* + 2. Let *z* be the number of men that must be eliminated in the second cycle in order to eliminate exactly $3m + 2$ people. We calculate *z* the following way:

$$z = 3m + 2 - d_1 = 3m + 2 - \frac{7m + 4 - a}{3} = \frac{2m + 2 + a}{3}.$$

Using our closed formula for α_j, we want to determine the value of α_{3z-a}. We start by calculating 3z – *a*.

$$3z - a = 3\left(\frac{2m + 2 + a}{3}\right) - a = 2m + 2.$$

So,

$$\alpha_{3z-a} = \frac{1}{2}\big(3(2m + 2) - 2\big) = 3m + 2.$$

Therefore, $J_{7m+4,3}(3m + 2) = 3m + 2$. □

Corollary 2.3: *If* $n = 7m + 2$ *and* $p = 3m + 1$ *or* $n = 7m + 4$ *and* $p = 3m + 2$ *then* $n = 2p + n \bmod p$.

3 Fixed Points in Round 3

Before the second cycle of elimination, we have

$$\alpha_j = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, \ldots\}.$$

Once d_1 people die, we are left with the following sequence if $n_1 \equiv 0 \bmod 3$

$$\beta_j = \{1, 2, 5, 7, 10, 11, 14, 16, 19, 20, \ldots\}.$$

This sequence can be written as the following piecewise function for $x \geq 0$.

$$\beta_j = \begin{cases} 9x+1 \text{ if } j = 4x+1 \\ 9x+2 \text{ if } j = 4x+2 \\ 9x+5 \text{ if } j = 4x+3 \\ 9x+7 \text{ if } j = 4x+4. \end{cases}$$

We will use this formula to prove theorems 3.1 through 3.4.

Theorem 3.1: *If m is a non-negative integer then* $J_{23m+7,3}(15m+5) = 15m+5$**.**

Proof: Let $n = 23m+7$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{23m+7}{3}.$$

So,

$$n_1 = n - d_1 = 23m + 7 - \frac{23m+7}{3} = \frac{46m+14}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$.

Let $a \in \{0,1,2\}$. Suppose $n_1 \equiv a \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1 - a}{3} = \frac{46m+4-3a}{9}.$$

Let z be the number of men that must be eliminated to eliminate $15m+5$ people. So,

$$z = 15m + 5 - d_1 - d_2 = \frac{20m+10+3a}{9}.$$

We need to determine the β_{3z-a} value using the sequence β_j. So we have,

$$3z - a = 3\left(\frac{20m+10+3a}{9}\right) - a = 4\left(\frac{5m+1}{3}\right) + 2.$$

Thus,

$$\beta_{3z-a} = 9\left(\frac{5m+1}{3}\right) + 2 = 15m+5.$$

Therefore, $J_{23m+7,3}(15m+5) = 15m+5$. □

Theorem 3.2: *If m is a non-negative integer then* $J_{23m+15,3}(15m+10) = 15m+10$**.**

Proof: Let $n = 23m+15$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{23m+15}{3}.$$

So,

$$n_1 = n - d_1 = 23m + 15 - \frac{23m+15}{3} = \frac{46m+30}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$.

Let $a \in \{0,1,2\}$. Suppose $n_1 \equiv a \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1 - a}{3} = \frac{46m+30-3a}{9}.$$

Let z be the number of men that must be eliminated to eliminate $15m+10$ people. So,

$$z = 15m + 10 - d_1 - d_2 = \frac{20m+15+3a}{9}.$$

We need to determine the β_{3z-a} value using the sequence β_j. So we have

$$3z - a = 3\left(\frac{20m+15+3a}{9}\right) - a = 4\left(\frac{5m+3}{3}\right) + 1.$$

Thus,

$$\beta_{3z-a} = 9\left(\frac{5m+3}{3}\right) + 1 = 15m + 10.$$

Therefore, $J_{23m+15,3}(15m+10) = 15m+10$. □

Theorem 3.3: *If m is a non-negative integer then* $J_{23m+21,3}(15m+14) = 15m+14$.

Proof: Let $n = 23m+21$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{23m+21}{3}.$$

So,

$$n_1 = n - d_1 = 23m + 21 - \frac{23m+21}{3} = \frac{46m+42}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$.

Let $a \in \{0,1,2\}$. Suppose $n_1 \equiv a \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1 - a}{3} = \frac{46m + 42 - 3a}{9}.$$

Let *z* be the number of men that must be eliminated to eliminate $15m + 14$ people. So,

$$z = 15m + 14 - d_1 - d_2 = \frac{20m + 21 + 3a}{9}.$$

We need to determine the β_{3z-a} value using the sequence β_j. So we have

$$3z - a = 3\left(\frac{20m + 21 + 3a}{9}\right) - a = 4\left(\frac{5m + 3}{3}\right) + 3.$$

Thus,

$$\beta_{3z-a} = 9\left(\frac{5m + 3}{3}\right) + 5 = 15m + 14.$$

Therefore, $J_{23m+21,3}(15m + 14) = 15m + 14$. □

Theorem 3.4: *If m is a non-negative integer then* $J_{23m+24,3}(15m + 16) = 15m + 16$.

Proof: Let $n = 23m + 24$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{23m + 24}{3}.$$

So,

$$n_1 = n - d_1 = 23m + 24 - \frac{23m + 24}{3} = \frac{46m + 48}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$.
Let $a \in \{0,1,2\}$. Suppose $n_1 \equiv a \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1 - a}{3} = \frac{46m + 48 - 3a}{9}.$$

Let *z* be the number of men that must be eliminated to eliminate $15m + 16$ people. So,

$$z = 15m + 16 - d_1 - d_2 = \frac{20m + 24 + 3a}{9}.$$

We need to determine the β_{3z-a} value using the sequence β_j. So we have

$$3z - a = 3\left(\frac{20m + 24 + 3a}{9}\right) - a = 4\left(\frac{5m + 3}{3}\right) + 4.$$

Thus,

$$\beta_{3z-a} = 9\left(\frac{5m+3}{3}\right) + 7 = 15m + 16 .$$

Therefore, $J_{23m+24,3}(15m+16) = 15m+16$. □

4 Fixed Points in Round 4

Before the third cycle of elimination we have for $n_1 \equiv 0 \bmod 3$

$$\beta_j = \{1, 2, 5, 7, 10, 11, 14, 16, 19, 20, \ldots\}.$$

Once d_2 people die, we are left with the following sequence for $n_2 \equiv 0 \bmod 3$

$$\chi_j = \{1, 2, 7, 10, 14, 16, 20, 23, 28, 29, 34, \ldots\}.$$

This sequence can be written as the following piecewise function for $x \geq 0$.

$$\chi_j = \begin{cases} 27x+1 \text{ if } j = 8x+1 \\ 27x+2 \text{ if } j = 8x+2 \\ 27x+7 \text{ if } j = 8x+3 \\ 27x+10 \text{ if } j = 8x+4 \\ 27x+14 \text{ if } j = 8x+5 \\ 27x+16 \text{ if } j = 8x+6 \\ 27x+20 \text{ if } j = 8x+7 \\ 27x+23 \text{ if } j = 8x+8. \end{cases}$$

We will use this formula to prove theorems 4.1 through 4.8.

Theorem 4.1: *If m is a non-negative integer then* $J_{73m+2,3}(57m+2) = 57m+2$.

Proof: Let $n = 73m+2$. We know that $n \equiv 0 \bmod 3, n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+2}{3}.$$

So,

$$n_1 = n - d_1 = 73m + 2 - \frac{73m+2}{3} = \frac{146m+4}{3}.$$

We know that $n_1 \equiv 0 \bmod 3, n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+4}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+4}{3} - \frac{73m+2}{9} = \frac{292m+8}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+8) - a}{3} = \frac{292m+8-9a}{27}.$$

Let z be the number of men that must be eliminated to eliminate $57m+2$ people. So,

$$z = 57m + 2 - d_1 - d_2 - d_3 = \frac{152m+16+9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z - a = 3\left(\frac{152m+16+9a}{27}\right) - a = 8\left(\frac{152m-56}{72}\right) + 8.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{152m-56}{72}\right) + 23 = 57m+2.$$

Therefore, $J_{73m+2,3}(57m+2) = 57m+2$. □

Theorem 4.2: *If m is a non-negative integer then* $J_{73m+6,3}(57m+5) = 57m+5$.

Proof: Let $n = 73m+6$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+6}{3}.$$

So,

$$n_1 = n - d_1 = 73m + 6 - \frac{73m+6}{3} = \frac{146m+12}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+12}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+12}{3} - \frac{146m+12}{9} = \frac{292m+24}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+24) - a}{3} = \frac{292m+24-9a}{27}.$$

Let z be the number of men that must be eliminated to eliminate $57m+5$ people. So,

$$z = 57m+5 - d_1 - d_2 - d_3 = \frac{152m+21+9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z - a = 3\left(\frac{152m+21+9a}{27}\right) - a = 8\left(\frac{19m-3}{9}\right) + 5.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{19m-3}{9}\right) + 14 = 57m+5.$$

Therefore, $J_{73m+6,3}(57m+5) = 57m+5$. □

Theorem 4.3: *If m is a non-negative integer then* $J_{73m+20,3}(57m+16) = 57m+16$.

Proof: Let $n = 73m+20$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+20}{3}.$$

So,

$$n_1 = n - d_1 = 73m+20 - \frac{73m+20}{3} = \frac{146m+40}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+40}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+40}{3} - \frac{146m+40}{9} = \frac{292m+80}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+80) - a}{3} = \frac{292m+80-9a}{27}.$$

Let z be the number of men that must be eliminated to eliminate $57m+16$ people. So,

$$z = 57m+16-d_1-d_2-d_3 = \frac{152m+52+9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z - a = 3\left(\frac{152m+52+9a}{27}\right) - a = 8\left(\frac{19m+2}{9}\right) + 4.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{19m+2}{9}\right) + 10 = 57m+16.$$

Therefore, $J_{73m+20,3}(57m+16) = 57m+16$. □

Theorem 4.4: *If m is a non-negative integer then* $J_{73m+24,3}(57m+19) = 57m+19$.

Proof: Let $n = 73m+24$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+24}{3}.$$

So,

$$n_1 = n - d_1 = 73m+24 - \frac{73m+24}{3} = \frac{146m+48}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+48}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+48}{3} - \frac{146m+48}{9} = \frac{292m+96}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+96) - a}{3} = \frac{292m + 96 - 9a}{27}.$$

Let *z* be the number of men that must be eliminated to eliminate $57m+19$ people. So,

$$z = 57m + 19 - d_1 - d_2 - d_3 = \frac{152m + 57 + 9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z - a = 3\left(\frac{152m + 57 + 9a}{27}\right) - a = 8\left(\frac{19m+6}{9}\right) + 1.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{19m+6}{9}\right) + 1 = 57m + 19.$$

Therefore, $J_{73m+24,3}(57m+19) = 57m+19$. □

Theorem 4.5: *If m is a non-negative integer then* $J_{73m+48,3}(57m+38) = 57m+38$.

Proof: Let $n = 73m + 48$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+48}{3}.$$

So,

$$n_1 = n - d_1 = 73m + 48 - \frac{73m+48}{3} = \frac{146m+96}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+96}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+96}{3} - \frac{146m+96}{9} = \frac{292m+192}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+192)-a}{3} = \frac{292m+192-9a}{27}.$$

Let z be the number of men that must be eliminated to eliminate $57m+38$ people. So,

$$z = 57m+38-d_1-d_2-d_3 = \frac{152m+114+9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z-a = 3\left(\frac{152m+114+9a}{27}\right) - a = 8\left(\frac{19m+12}{9}\right)+2.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{19m+12}{9}\right)+2 = 57m+38.$$

Therefore, $J_{73m+48,3}(57m+38) = 57m+38$. □

Theorem 4.6: *If m is a non-negative integer then* $J_{73m+52,3}(57m+41) = 57m+41$.

Proof: Let $n = 73m+52$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+52}{3}.$$

So,

$$n_1 = n - d_1 = 73m+52-\frac{73m+52}{3} = \frac{146m+104}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+104}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+104}{3} - \frac{146m+104}{9} = \frac{292m+208}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+208) - a}{3} = \frac{292m+208-9a}{27}.$$

Let *z* be the number of men that must be eliminated to eliminate $57m+41$ people. So,

$$z = 57m + 41 - d_1 - d_2 - d_3 = \frac{152m+119+9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z - a = 3\left(\frac{152m+119+9a}{27}\right) - a = 8\left(\frac{19m+7}{9}\right) + 7.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{19m+7}{9}\right) + 20 = 57m+41.$$

Therefore, $J_{73m+52,3}(57m+41) = 57m+41$. □

Theorem 4.7: *If m is a non-negative integer then* $J_{73m+66,3}(57m+52) = 57m+52$.

Proof: Let $n = 73m+66$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+66}{3}.$$

So,

$$n_1 = n - d_1 = 73m + 66 - \frac{73m+66}{3} = \frac{146m+132}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+132}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+132}{3} - \frac{146m+132}{9} = \frac{292m+264}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+264) - a}{3} = \frac{292m+264-9a}{27}.$$

Let z be the number of men that must be eliminated to eliminate $57m+52$ people. So,

$$z = 57m+52 - d_1 - d_2 - d_3 = \frac{152m+150+9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z - a = 3\left(\frac{152m+150+9a}{27}\right) - a = 8\left(\frac{19m+12}{9}\right) + 6.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{19m+12}{9}\right) + 16 = 57m+52.$$

Therefore $J_{73m+66,3}(57m+52) = 57m+52$. □

Theorem 4.8: *If m is a non-negative integer then* $J_{73m+70,3}(57m+55) = 57m+55$.

Proof: Let $n = 73m+70$. We know that $n \equiv 0 \bmod 3$, $n \equiv 1 \bmod 3$ or $n \equiv 2 \bmod 3$. The following is the proof of the case when $n \equiv 0 \bmod 3$. The other two cases follow similarly and are left to the diligent reader.

Case 1: Suppose $n \equiv 0 \bmod 3$, then

$$d_1 = \left\lfloor \frac{n}{3} \right\rfloor = \frac{n}{3} = \frac{73m+70}{3}.$$

So,

$$n_1 = n - d_1 = 73m+70 - \frac{73m+70}{3} = \frac{146m+140}{3}.$$

We know that $n_1 \equiv 0 \bmod 3$, $n_1 \equiv 1 \bmod 3$ or $n_1 \equiv 2 \bmod 3$. The following is the proof of the subcase when $n_1 \equiv 0 \bmod 3$. The other two subcases follow similarly and are left to the reader.

Subcase 1: Suppose $n_1 \equiv 0 \bmod 3$, then

$$d_2 = \left\lfloor \frac{n_1}{3} \right\rfloor = \frac{n_1}{3} = \frac{146m+140}{9}.$$

So,

$$n_2 = n_1 - d_2 = \frac{146m+140}{3} - \frac{146m+140}{9} = \frac{292m+280}{9}.$$

Let $a \in \{0,1,2\}$. Suppose $n_2 \equiv a \bmod 3$, then

$$d_3 = \left\lfloor \frac{n_2}{3} \right\rfloor = \frac{n_2 - a}{3} = \frac{\frac{1}{9}(292m+280) - a}{3} = \frac{292m+280-9a}{27}.$$

Let z be the number of men that must be eliminated to eliminate $57m+55$ people. So,

$$z = 57m+55-d_1-d_2-d_3 = \frac{152m+155+9a}{27}.$$

We need to determine the χ_{3z-a} value using the sequence χ_j. So we have

$$3z - a = 3\left(\frac{152m+155+9a}{27}\right) - a = 8\left(\frac{19m+16}{9}\right) + 3.$$

Thus,

$$\chi_{3z-a} = 27\left(\frac{19m+16}{9}\right) + 7 = 57m+55.$$

Therefore, $J_{73m+70,3}(57m+55) = 57m+55$. □

5 Conclusion

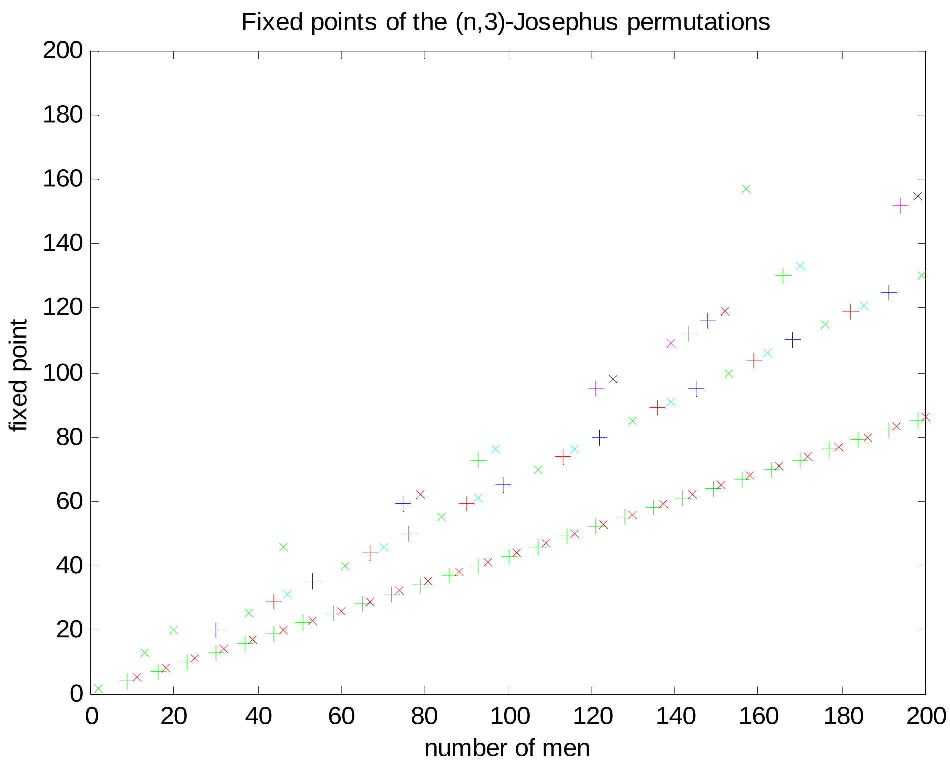
In this paper, we discovered and proved the following patterns of each round where *m* is a non-negative integer:

Table 6

Round	Pattern
1	none
2	$J_{7m+2,3}(3m+1) = 3m+1$ $J_{7m+4,3}(3m+2) = 3m+2$
3	$J_{23m+7,3}(15m+5) = 15m+5$ $J_{23m+15,3}(15m+10) = 15m+10$ $J_{23m+21,3}(15m+14) = 15m+14$ $J_{23m+24,3}(15m+16) = 15m+16$
4	$J_{73m+2,3}(57m+2) = 57m+2$ $J_{73m+6,3}(57m+5) = 57m+5$ $J_{73m+20,3}(57m+16) = 57m+16$ $J_{73m+24,3}(57m+19) = 57m+19$ $J_{73m+48,3}(57m+38) = 57m+38$

	$J_{73m+52,3}(57m+41) = 57m+41$ $J_{73m+66,3}(57m+52) = 57m+52$ $J_{73m+70,3}(57m+55) = 57m+55$

Using these formulas for the fixed points of rounds two, three and four of the (*n*, 3)-Josephus permutations, we arrive at a final picture of the visible pattern these relationships create. Below is a graph of the fixed points of the (*n*, 3)-Josephus permutations we identified.



6 Further Research

1. In the $J(n,3)$ Josephus permutations, there are some interesting fixed points in which person n is the n^{th} person to die. That is, $J_{n,3}(n) = n$. For example, $J_{13,3}(13) = 13$ and $J_{46,3}(46) = 46$. The following table lists some of these special fixed points.

Table 7

n	2	13	20	46	157	236	532	1198	4045
$J_{n,3}(n)$	2	13	20	46	157	236	532	1198	4045

This pattern of fixed points appears exponential in nature. An investigation of this may be fruitful.

2. Patterns similar to the fixed points of the $(n,3)$-Josephus Dilemma occur in the $(n,4)$-Josephus Dilemma. To illustrate this thought, the patterns of fixed points in the second round are named and proven below in theorems 6.1 through 6.3.

Beginning with the sequence

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,...\}$$

we are left with the following sequence after the first cycle of elimination when $k = 4$.

$$\Delta = \{1, 2, 3, 5, 7, 9, 10, 11, 13, 14, 15, 17,...\}.$$

This sequence can be written as the following piecewise function.

$$\Delta_j = \begin{cases} \frac{1}{3}(4j-1) \text{ if } j = 3m+1 \\ \frac{1}{3}(4j-2) \text{ if } j = 3m+2 \\ \frac{1}{3}(4j-3) \text{ if } j = 3m+3 \end{cases}$$

We will use this formula to prove theorems 6.1 through 6.4. Note that no multiple of 4 is present in this sequence, therefore no multiple of 4 can be a fixed point. This proof is similar to that of Lemma 1.6.

Theorem 6.1: *If m is a non-negative integer then* $J_{13m+3,4}(4m+1) = 4m+1$.

Proof: Let $n = 13m+3$. We know that $n \equiv 0 \bmod 4$, $n \equiv 1 \bmod 4$, $n \equiv 2 \bmod 4$, or $n \equiv 3 \bmod 4$. Let $a \in \{0,1,2,3\}$. Suppose $n \equiv a \bmod 4$, then

$$d_1 = \left\lfloor \frac{n}{4} \right\rfloor = \frac{n-a}{4} = \frac{13m+3-a}{4}.$$

Let *z* be the number of men that must be eliminated to eliminate $4m+1$ people. So,

$$z = 4m+1-d_1 = \frac{3m+1+a}{4}.$$

We need to determine the Δ_{4z-a} value using the sequence Δ_j. So, we have

$$4z-a = 4\left(\frac{3m+1+a}{4}\right) - a = 3m+1.$$

Thus,

$$\Delta_{4z-a} = \frac{4(3m+1)-1}{3} = 4m+1.$$

Therefore, $J_{13m+3,4}(4m+1) = 4m+1$. □

Theorem 6.2: *If m is a non-negative integer then* $J_{13m+6,4}(4m+2) = 4m+2$.

Proof: Let $n = 13m+6$. We know that $n \equiv 0 \bmod 4$, $n \equiv 1 \bmod 4$, $n \equiv 2 \bmod 4$, or $n \equiv 3 \bmod 4$. Let $a \in \{0,1,2,3\}$. Suppose $n \equiv a \bmod 4$, then

$$d_1 = \left\lfloor \frac{n}{4} \right\rfloor = \frac{n-a}{4} = \frac{13m+6-a}{4}.$$

Let z be the number of men that must be eliminated to eliminate $4m+2$ people. So,

$$z = 4m+2-d_1 = \frac{3m+2+a}{4}.$$

We need to determine the Δ_{4z-a} value using the sequence Δ_j. So, we have

$$4z-a = 4\left(\frac{3m+2+a}{4}\right) - a = 3m+2.$$

Thus,

$$\Delta_{4z-a} = \frac{4(3m+2)-2}{3} = 4m+2.$$

Therefore, $J_{13m+6,4}(4m+2) = 4m+2$. □

Theorem 6.3: *If m is a non-negative integer then* $J_{13m+9,4}(4m+3) = 4m+3$.

Proof: Let $n = 13m+9$. We know that $n \equiv 0 \bmod 4$, $n \equiv 1 \bmod 4$, $n \equiv 2 \bmod 4$, or $n \equiv 3 \bmod 4$. Let $a \in \{0,1,2,3\}$. Suppose $n \equiv a \bmod 4$, then

$$d_1 = \left\lfloor \frac{n}{4} \right\rfloor = \frac{n-a}{4} = \frac{13m+9-a}{4}.$$

Let z be the number of men that must be eliminated to eliminate $4m+3$ people. So,

$$z = 4m+3-d_1 = \frac{3m+3+a}{4}.$$

We need to determine the Δ_{4z-a} value using the sequence Δ_j. So, we have

$$4z-a = 4\left(\frac{3m+3+a}{4}\right) - a = 3m+3.$$

Thus,

$$\Delta_{4z-a} = \frac{4(3m+3)-3}{3} = 4m+3.$$

Therefore, $J_{13m+9,4}(4m+3) = 4m+3$. □

Appendix

This is a maple program that calculates the order of elimination of *n* people with a skip factor of 3. The algorithm is adopted from [3].

```
> m:=3^4:L:=m:                          # number of people
> J:=array(1..L):f:=array(1..L):        # tells data types f=working list
> for j from 1 to L do f[j]:=j od:      # makes f=[1,2,3,...,m]
> for n from 1 to m-1 do                # J DATA LOOP
> x1:=f[1]: x2:=f[2]:                   # temporary variables
> for i from 1 to L-2 do                # SHIFTING sub-LOOP
> f[i]:=f[i+2] od:                      # moves guys in list f 2 units left
> f[L-1]:=x1: f[L]:=x2:                 # moves guys 1 and 2 to end of list
> J[n]:=f[1]:                           # person to kill is now the first in list
> if J[n]=n then print(m,n) fi;         # check to see if this guy is fixed point
> for k from 1 to L-1 do                # KILLING sub-sub-LOOP
> f[k]:=f[k+1] od:                      # shifts person to kill to end of list
> f[L]:=0: L:=L-1: od:                  # sets dead guy=0; end of J DATA LOOP
> J[m]:=f[1]:                           # last person alive is now first in list
> if J[m]=m then print(m,m) fi;         # is last person fixed point?
> print(J);                             # prints order of death
```


Acknowledgements

We thank the National Security Agency and National Science Foundation for funding our research at Miami University in Oxford, Ohio. We also thank the department of Mathematics and Statistics of Miami University involved in the SUMSRI program, especially Bonita Porter for her dedication in coordinating the program. A special thanks is due to Dr. Chawne Kimber and Angela Grant for their kind encouragement and continual mentoring. This project could not have been completed without them. We thank our family and friends for their lifelong gifts of generosity and loving support whose presence in our lives brought us to where we are today. Finally, we thank God for His guidance and amazing love.

References

[1] Cook, Ensley, Shehy, *Josephus and the Amazing Technicolor Fixed Points*, Shippensburg University Technical Report, 1997. http://www.ship.edu/~deensl/pgss/Joseph/

[2] L. Euler, "Observationes circa novum et singulare progressiounum genus," Opera Omnia Series Prima, Opera Mathematica 7 (MCMXXIII) 246-261.

[3] Graham, Knuth, & Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd edition, Addison-Wesley: Reading, 1994.

[4] Herstein& Kaplansky, *Matters Mathematical*, Chelsea: New York, 1978.

[5] Josephus, *Josephus: The Complete Works*, (William Whiston, A. M., translator), Nelson: Nashville, 1998.